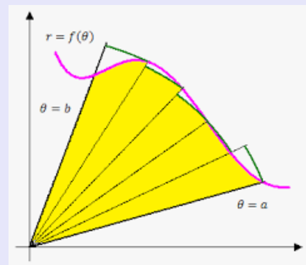
Calculus II

Lecture 17



Feb 19-8:47 AM

Class QZ 14

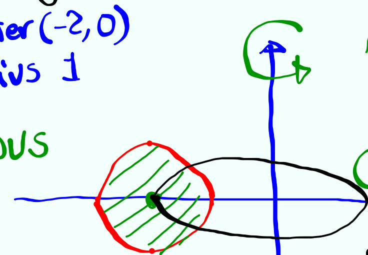
Find the volume of the solid obtained by rotating the circle $(x+2)^2 + y^2 = 1$ about the y-axis.

Exact Ans. only

Center $(-2,0)$
Radius 1

$A = \pi r^2 = \pi(1)^2 = \boxed{\pi}$

By theorem of Pappus



$(\bar{x},\bar{y}) = (-2,0)$

$V = A \cdot$ distance traveled by its centroid about axis of Rev.

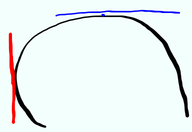
$d = 2\pi|\bar{x}|$
$= 2\pi|-2| = 4\pi$

$V = \pi \cdot 4\pi$
$= \boxed{4\pi^2}$

Jul 10-7:14 AM

Find points on the graph of $x = t^3 - 3t$, $y = t^3 - 3t^2$ where the tan. line is horizontal or vertical.

Horzontal tan. line $\rightarrow m = 0$



Vertical tangent line
m is undefined.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 6t}{3t^2 - 3}$$

Horizontal tan. line

$\frac{dy}{dt} = 0$ & $\frac{dx}{dt} \neq 0$

$3t^2 - 6t = 0 \quad 3t(t-2) = 0$

$t = 0 \rightarrow (0,0)$

$t = 2 \rightarrow (2,-4)$

$\frac{dx}{dt} = 0$ & $\frac{dy}{dt} \neq 0$

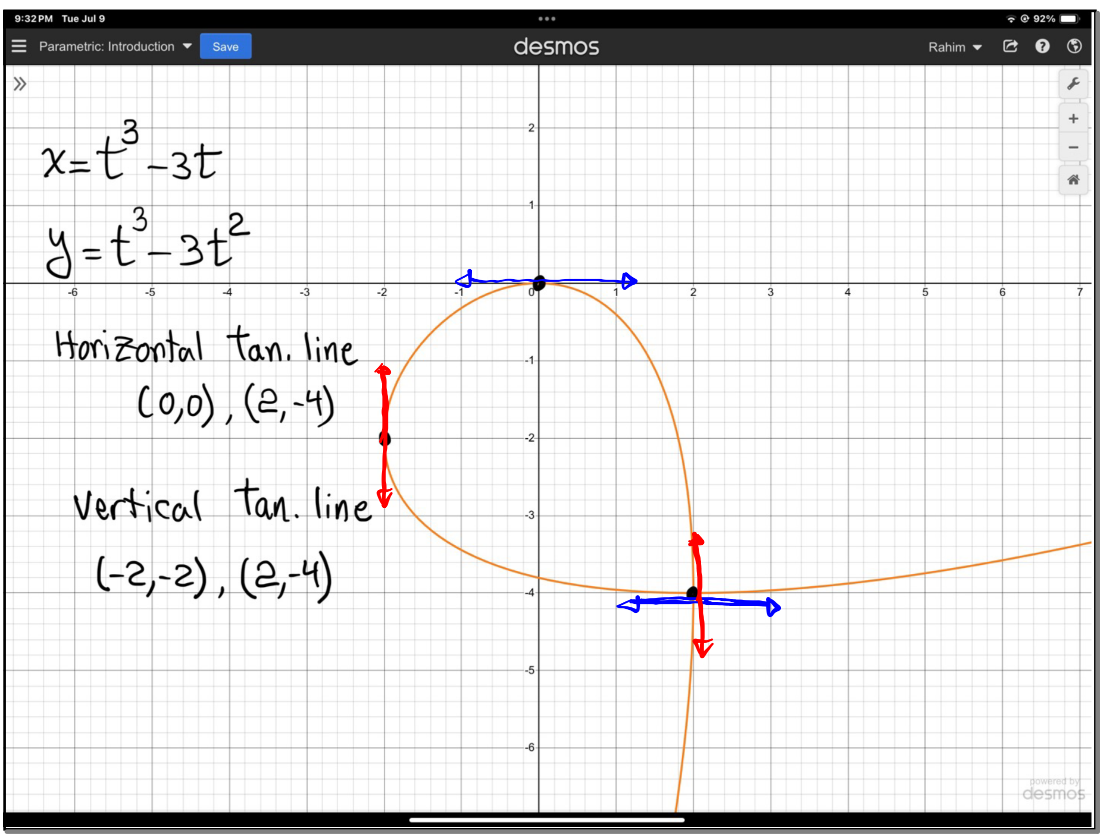
$3t^2 - 3 = 0 \quad 3(t^2 - 1) = 0$

$t = \pm 1$

$t = 1 \rightarrow (-2,-2)$

$t = -1 \rightarrow (2,-4)$

Jul 9-8:42 AM



Jul 10-6:57 AM

Consider the curve given by $x = t^2$ and $y = t^3$. $y \in \mathbb{R}$ $(-\infty, \infty)$ $x \geq 0$ $[0, \infty)$

1) Eliminate the parameter and find equation in rectangular Coordinates.

$y^2 = (t^3)^2$ $\quad y^2 = t^6$ $\Rightarrow \boxed{y^2 = x^3}$

$x^3 = (t^2)^3$ $\quad x^3 = t^6$

2) Discuss domain & Range

Domain $[0, \infty)$

Range $(-\infty, \infty)$

$\frac{d}{dx}[y^2] = \frac{d}{dx}[x^3]$

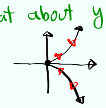
$2y \frac{dy}{dx} = 3x^2 \rightarrow \frac{dy}{dx} = \frac{3x^2}{2y}$

3) Discuss increasing & Decreasing as well as Vertical & Horizontal tangent lines.

Inc. $y > 0$

Dec. $y < 0$

what about $y = 0$? $x = 0 \rightarrow (0,0)$



$\frac{dy}{dx} = \frac{0}{0}$ I.F.

$(0,0)$ is a singular Pt.

4) Discuss concavity

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2}{2t} = \frac{3}{2}t,\ t \neq 0$

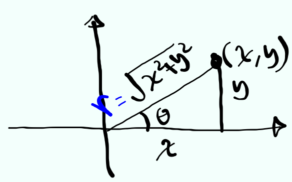
$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{3}{2}t\right]}{2t} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t}$

If $t > 0$ $\quad \frac{d^2y}{dx^2} > 0 \rightarrow$ C.U.

If $t < 0$ $\quad \frac{d^2y}{dx^2} < 0 \rightarrow$ C.D.

Jul 8-10:56 AM

Back to Polar



$r^2 = x^2 + y^2$

Polar form

(r, θ)

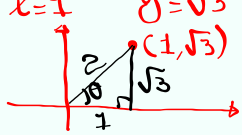
$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$

$\tan\theta = \frac{y}{x}$

Convert $(1, \sqrt{3})$ to Polar.

$x = 1$ $\quad y = \sqrt{3}$



$r^2 = x^2 + y^2$ $\quad r^2 = 1^2 + (\sqrt{3})^2$

$r^2 = 4$ $\quad r = 2$

$x = r\cos\theta \longrightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\cos\theta = \frac{1}{2}$ $\quad \theta = \frac{\pi}{3}$

$\sin\theta = \frac{\sqrt{3}}{2}$ $\quad \theta = \frac{\pi}{3}$

$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$ $\quad \theta = \frac{\pi}{3}$

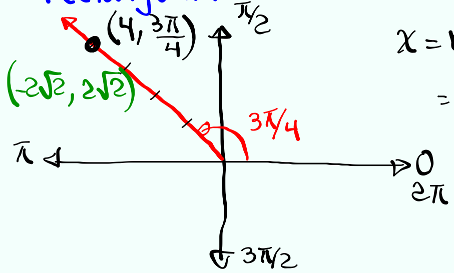
$(1, \sqrt{3})$

$\left(2, \frac{\pi}{3}\right)$

Jul 10-8:43 AM

Plot polar point $\left(4, \frac{3\pi}{4}\right)$, convert to rectangular.

$r=4$ $\theta=\frac{3\pi}{4}$

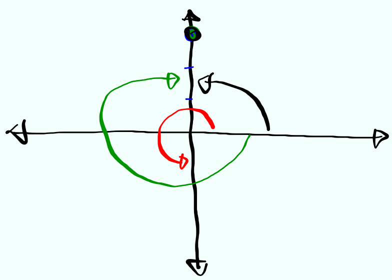


$x = r\cos\theta$

$= 4\cdot\cos\frac{3\pi}{4} = 4\cdot -\frac{\sqrt{2}}{2} = -2\sqrt{2}$

$y = r\sin\theta = 4\cdot\sin\frac{3\pi}{4} = 2\sqrt{2}$

Plot polar Points $\left(-3, \frac{3\pi}{2}\right)$, $\left(3, -\frac{3\pi}{2}\right)$, $\left(3, \frac{\pi}{2}\right)$



Jul 10-8:50 AM

Graph $r=3$, $r\cos\theta = 3$ $\quad$ $r\sin\theta = -3$

$x^2+y^2=r^2$

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = 3^2$

$\rightarrow x^2+y^2=3^2$

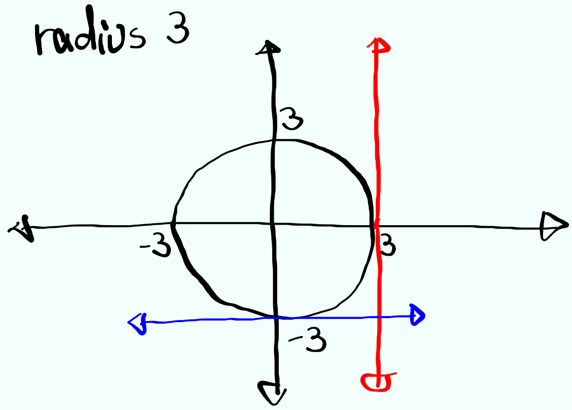
Circle at $(0,0)$ radius 3

$\rightarrow x=3$

Vertical line

$\rightarrow y=-3$

Horizontal line

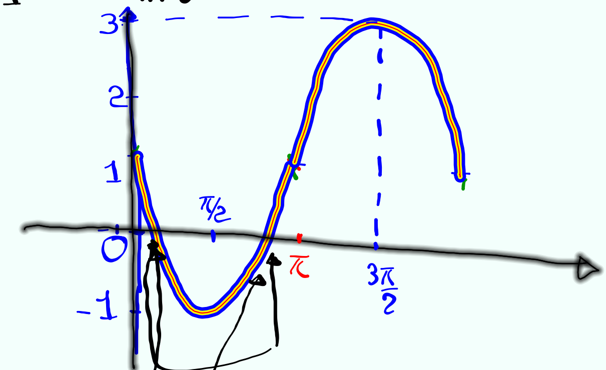


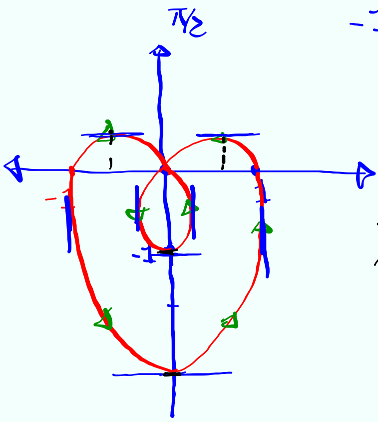
Jul 10-8:59 AM

Graph $r = 1 - 2\sin\theta$

Graph

$y = 1 - 2\sin x$





$\frac{\pi}{6}$ $\quad \frac{5\pi}{6}$

x-Ints $\rightarrow y = 0$

$1 - 2\sin x = 0$

$\sin x = \frac{1}{2}$

Ref. Angle $\frac{\pi}{6}$

Jul 10-9:03 AM

$r = 1 - 2\sin\theta$

$r = f(\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}[r\sin\theta]}{\frac{d}{d\theta}[r\cos\theta]} = \frac{\frac{dr}{d\theta}\cdot\sin\theta + r\cdot\cos\theta}{\frac{dr}{d\theta}\cdot\cos\theta + r\cdot(-\sin\theta)}$$

$$= \frac{-2\cos\theta\cdot\sin\theta + (1-2\sin\theta)\cdot\cos\theta}{-2\cos\theta\cdot\cos\theta - (1-2\sin\theta)\cdot\sin\theta}$$

$$= \frac{-4\cos\theta\sin\theta + \cos\theta}{-2\cos^2\theta + 2\sin^2\theta - \sin\theta}$$

Horizontal tan. line

$\frac{dy}{d\theta} = 0, \ \frac{dx}{d\theta} \neq 0$

$-4\cos\theta\sin\theta + \cos\theta = 0$

$\cos\theta[-4\sin\theta + 1] = 0$

$\cos\theta = 0 \qquad \sin\theta = \frac{1}{4}$

$\theta = \pi/2, \frac{3\pi}{2} \qquad \theta = \sin^{-1}\frac{1}{4}$

$\theta \approx 14^\circ, 180^\circ - 14^\circ$

$\approx 166^\circ$

Vertical tan. line

$\frac{dx}{d\theta} = 0, \ \frac{dy}{d\theta} \neq 0$

$-2\cos^2\theta + 2\sin^2\theta - \sin\theta = 0$

$-2(1 - \sin^2\theta) + 2\sin^2\theta - \sin\theta = 0$

$\underset{a}{4\sin^2\theta} - \underset{b}{\sin\theta} \underset{c}{- 2} = 0$ $\quad$ Q-formula $\rightarrow$ $\sin\theta = \frac{1 \pm \sqrt{33}}{8}$

Jul 10-9:14 AM

$$\text{Sin}\,\theta = \frac{1+\sqrt{33}}{8}$$

$\text{Sin}\,\theta \approx .8$ $\quad \theta = \text{Sin}^{-1}(.8)$ $\quad$ Ref. angle $\approx 53^\circ$

$\boxed{53^\circ}$ & $180^\circ - 53^\circ = \boxed{127^\circ}$

$$\text{Sin}\,\theta = \frac{1-\sqrt{33}}{8}$$

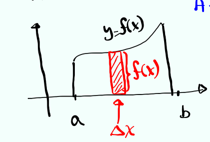
Ref. Angle $\text{Sin}^{-1}(.6) \approx 37^\circ$

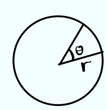
$\text{Sin}\,\theta \approx -.6$ $\quad$ QIII $\quad \theta = 180^\circ + 37^\circ = \boxed{217^\circ}$

QIV $\quad \theta = 360^\circ - 37^\circ \approx \boxed{323^\circ}$

Jul 10-9:29 AM

Find the area $\quad A = \int_a^b f(x)\,dx$



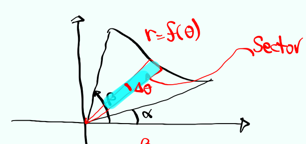


$A = \frac{1}{2} r^2 \theta$

If $\theta = 2\pi$

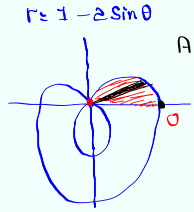
$A = \frac{1}{2} r^2 \cdot 2\pi$

$= \pi r^2$



$$\text{Area} = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta$$

$r = 1 - 2\,\text{Sin}\,\theta$



$$A = \int_0^{\pi/6} \frac{1}{2}(1 - 2\text{Sin}\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} [1 - 4\text{Sin}\theta + 4\text{Sin}^2\theta]\,d\theta$$

$$= \frac{1}{2}\left[\theta - 4\cdot -\text{Cos}\theta + 4\int \frac{1-\text{Cos}2\theta}{2}\,d\theta\right]\Big|_0^{\pi/6}$$

$$= \frac{1}{2}\left[\theta + 4\text{Cos}\theta + 2\left(\theta - \frac{1}{2}\text{Sin}2\theta\right)\right]\Big|_0^{\pi/6}$$

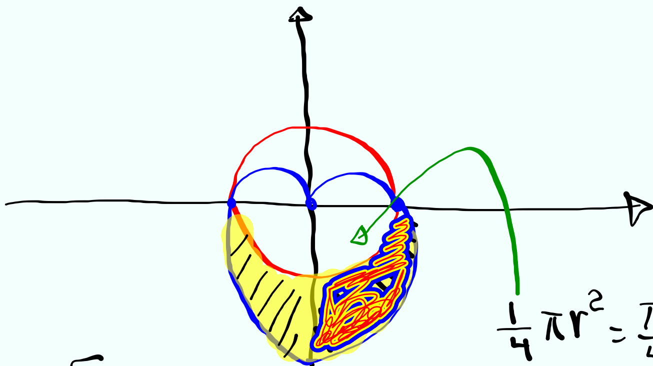
$$= \frac{1}{2}\left[3\theta + 4\text{Cos}\theta - \text{Sin}2\theta\right]\Big|_0^{\pi/6} = \boxed{}$$

Jul 10-9:44 AM

Find the area inside of $r = 1 - \sin\theta$ and outside of $r = 1$.

$r = 1 - \sin\theta$

θ	r
0	1
$\frac{\pi}{2}$	0
π	1
$\frac{3\pi}{2}$	2
2π	1



$\frac{1}{4}\pi r^2 = \frac{\pi}{4}$

$$A = 2\left[\int_{3\pi/2}^{2\pi} \frac{1}{2}(1 - \sin\theta)^2\, d\theta - \frac{\pi}{4}\right] = \int_{3\pi/2}^{2\pi} (1 - 2\sin\theta + \sin^2\theta)\, d\theta - \frac{\pi}{2}$$

Jul 10-9:56 AM

Find the area bounded by $r = \cos\theta$

$r = \cos\theta$

$r^2 = r\cos\theta$

$x^2 + y^2 = x$

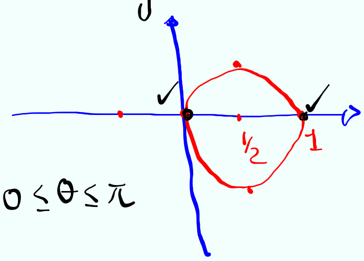
$x^2 - x + \frac{1}{4} + y^2 = 0 + \frac{1}{4}$

$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$

Circle center $\left(\frac{1}{2}, 0\right)$, radius $\frac{1}{2}$

$A = \pi r^2 = \pi \cdot \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$ ✓

θ	r
0	1
$\frac{\pi}{2}$	0
π	-1



$0 \le \theta \le \pi$

$$A = \int_0^{\pi} \frac{1}{2} r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \cos^2\theta\, d\theta$$

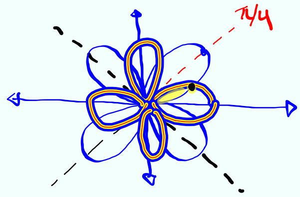
$$= \frac{1}{2}\int_0^{\pi} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]\Big|_0^{\pi}$$

$$= \frac{1}{4}\pi + 0 = \boxed{\pi/4}$$

Jul 10-10:05 AM

Find the area that lies inside of both curves $r=\sin 2\theta$ & $r=\cos 2\theta$



θ	r
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	0

Ans: $\frac{1}{2}\pi - 1$

θ	r
0	1
$\frac{\pi}{4}$	0

$\sin 2\theta = \cos 2\theta$

$\frac{\sin 2\theta}{\cos 2\theta} = 1$

Ref. angle $\frac{\pi}{4}$

$\tan 2\theta = 1$ $\quad 2\theta = \pi/4 \quad \theta = \frac{\pi}{8}$

$A = 16\int_0^{\pi/8} \frac{1}{2}(\sin 2\theta)^2\, d\theta$

$$A = 8\int_0^{\pi/8} \sin^2 2\theta \, d\theta = 8\int_0^{\pi/8} \frac{1-\cos 4\theta}{2}\, d\theta$$

$$= 4\left[\theta - \frac{1}{4}\sin 4\theta\right]\Big|_0^{\pi/8} = 4\left[\frac{\pi}{8} - \frac{1}{4}\sin\frac{4\pi}{8} - 0\right]$$

$$= 4\left[\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right]$$

$$= \boxed{\frac{\pi}{2} - 1}$$